

## Theory to Practice

### Article

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### Betting on the Future with a Cloudy Crystal Ball? Revenue Forecasting, Financial Theory, and Budgets— An Expanded Treatment

**Abstract:** *Accurately predicting revenue growth is nearly impossible. Predicting the peaks and valleys of the business cycle is even more hopeless. This matters because tax revenues are largely driven by economic growth. Volatile, unpredictable revenue growth causes all sorts of unpleasant responses on the part of governments, most commonly manic-depressive patterns of spending and taxing. Fortunately, modern financial economics gives us a set of tools that can be used to manage volatility. These tools include tax portfolio analyses, hedging and buffering strategies, and, in the context of present-value balance, consumption smoothing based on sustainable spending rules. These tools are based on mean-variance analysis, analysis of covariance, the use of stochastic processes to model movements in financial variables, and optimal control theory to formulate solutions to those processes. This article shows how these tools can be used to inform fiscal decision-making. Our focus is on state governments, but our analysis applies to all juris-*

*dictions that face a hard budget constraint and must therefore balance spending increases against revenue growth.*

Prediction is a mug's game. The further governments look into their financial futures, the murkier the horizon. Instead of asking questions like, "How much will revenue grow?," we ought to ask the question, "Given that we can't predict the future, how can we get a good result no matter what the economy throws at us?"

In this extended e-version of our September/October article in *PAR*, we show how advances in financial theory can help us answer this question. The American Finance Association started the same year as ASPA. Its official organ, the *Journal of Finance*, began in 1947, seven years after *PAR*. Contemporary financial theory is all about understanding and making prudent choices in the face of financial uncertainty. It provides fairly well codified answers to questions such as: how much can we expect revenue to

grow, how much revenue volatility can we face, how much can we reduce volatility without reducing growth expectations, and how long can we wait before acting? In answering these questions, financial analysts have reformulated and quantified old concepts such as diversification, insurance, options, and futures contracts and suggested some new ways of using them to manage risk.

In the course of introducing a toolkit that ought to be part of the standard repertoire of public sector financial managers, we offer several lessons for general readers. We also set out the mathematical and statistical procedures that are needed to use these tools—mean-variance analysis, covariance analysis, and stochastic process analysis—and some examples of their use based upon budgetary and financial data from the State of Oregon.

### **Growth Analysis**

Rapid and sustained revenue growth tends to encourage unsustainable spending increases or tax cuts. When recession strikes, governments, which face a "hard budget constraint"<sup>1</sup> and cannot practice discretionary monetary and fiscal policy, engage in a variety of expedients to cope with the emergency (Cornia, Nelson, and Wilko 2004; Hou 2006). Many of them, however, are quite wasteful. These expedients include cutting maintenance or deferring replacement of assets, raiding trust funds (which is costly because it implicitly means borrowing at the taxable rate rather than explicitly at the tax exempt rate, which is usually much lower), or shifting fiscal obligations to local governments. As long as recession is fresh in the minds of public officials, their control of the purse strings remains tight. Gradually, however, funds accumulate and the need to spend becomes overriding. Consequently, the process is repeated over and over again:

spending increases and tax cuts add force to booms and spending cuts and tax increases deepen cyclical troughs (Levinson 1998; Poterba 1995).

The problem faced by budget makers derives primarily from volatility in revenue growth. The fact is that we cannot accurately predict revenue growth from one year to the next or the timing of the business cycle. What we can do is quantify expectations in terms of average growth rates and actuarial volatility. The technical term for actuarial volatility is variance. Variance is the distance from an average or mean, and in this instance the mean we are talking about is the expected rate of revenue growth.<sup>2</sup>

Where the distribution of growth rates is stable over time and periodic growth rates are independent of one another, the simple average of past growth rates or the arithmetic mean is the best estimator of expected growth rates. The arithmetic mean should be distinguished from the geometric mean, which measures the compound rate of growth over several periods. The geometric mean will always be less than the arithmetic mean, except where the rate of growth is constant. Moreover, the gap between the two will widen as variance in the growth rate increases. In fact, for large samples, the geometric mean will be approximately equal to the arithmetic mean, less one-half of the variance of actual growth rates. The variance is the simple average of the squared differences between actual growth rates and the mean growth rate—or mean squared error. The standard deviation of the distribution ( $\sigma$ ) is simply the square root of the variance.

***Lesson 1: The simple average of past revenue growth rates is the best estimator of expected growth rates.***

Mean-variance analysis starts with the same steps taken by sophisticated revenue forecasters (Bowerman, O'Connell, and Koehler 2005; Makridakis, Wheelwright, and Hyndman 1998)—with a multiplicative decomposition of historical data into components: the trend or rate of long-term growth or decline; a cycle or regular, periodic oscillation in growth rates; and a random component. The trend is simply the mean growth rate and the other two components, the variance. The problem with the standard approach to revenue forecasting is that, while there is a large cyclical component to revenue growth *ex post* due to a relationship with an underlying dominant factor, growth in the economy as a whole, there is no consistent cyclical component *ex ante*.

However, knowledge that revenue growth is driven in part by an underlying dominant factor allows us to decompose revenue volatility into an *unsystematic* component and a *systematic* component. The former is entirely random and can be managed, in part, through diversification. The latter reflects changes in the underlying economy and can be managed, at a cost, through hedging and buffering. Moreover, by analogy, an understanding of hedging also helps us to understand the benefits and costs of participation in self-insurance pools and rainy day funds. Diversification is the focus of the next section of this essay and hedging the third.

### **Using Portfolio Theory to Manage Unsystematic Volatility**

Effective diversification of revenue sources can reduce revenue volatility without also reducing revenue growth. However, diversification depends less on the number of tax types used than it does on the way they perform relative to one another (i.e., their covariance). An appreciation of how diversification works and can be used is one of the main ideas tax specialists have taken from

modern financial theory in recent years. Indeed, covariance analysis and its extension, portfolio theory, are conceptually and mathematically similar to some traditional elements of public finance. These include analysis of vertical and horizontal equity (which is usually done with cross-sectional data on personal income and tax payments), and revenue elasticity analysis (which is usually done with time-series or panel data on personal income and tax yield).

The difference is that portfolio theory looks at how various tax types incorporating different design elements covary with each other and with the underlying economy. Its aim is to understand how a portfolio of tax types can be used to manage volatility in revenue growth. In contrast, tax specialists once simply asserted an inherent tradeoff between revenue growth and volatility (Groves and Kahn 1952). According to this view, income-elastic tax bases tend to grow faster than the economy, but fluctuations in income over the business cycle cause them to be unstable. In contrast, inelastic tax bases grow more slowly or not at all, but are highly stable. This view has a basis in fact. Governments that rely heavily on highly progressive, comprehensive income taxes have both high growth rates and high revenue volatility. The state of Oregon illustrates this phenomenon. From 1950-2000, real revenue growth averaged between 3.5 and 4 percent per annum (nominal revenue growth rate for the whole period was almost 9 percent). Revenue volatility (the variance, expressed as the standard deviation, or  $\sigma$ , of year-on-year revenue increments) was a whopping 7.6 percent (13.4 percent for nominal revenues).

In the meantime, empirical analysis has demonstrated that the tradeoff between revenue growth and stability is not as acute as was once thought. Income taxes are not

necessarily more volatile or faster growing than broad-based consumption taxes; corporate income taxes grow more slowly than personal income taxes and are more volatile; even some specific excises, like motor fuel taxes, are fast growing and some are quite volatile (Bruce, Fox, and Tuttle 2006; Dye and Merriman 2004; Otsuka and Braun 1999; Sobel and Holcombe 1996).

Indeed, Donald Bruce, William Fox, and M. H. (2004; see also Fox 2003) show that the composition of the tax base, rate structures, and elements of administration can have bigger effects on volatility and growth than tax type. For example, state policymakers can often significantly lower the volatility of revenue from broad-based taxes without adversely affecting revenue growth simply by eliminating exemptions or equalizing marginal rates; i.e., reducing the progressivity of the overall tax structure.

***Lesson 2: Governments cannot significantly reduce volatility in revenue growth by substituting one tax type for another (e.g., a broad-based goods and services tax for an income tax or vice versa) unless they also are willing to replace a more progressive tax with a less progressive tax.***

What this conclusion omits is that many governments can reduce volatility by diversifying their tax portfolios, without reducing the progressivity of their tax base. How does

diversification of tax portfolios work? The answer is that the volatility of a tax portfolio depends upon the covariance of its components, not on the average of its individual tax types. Hence, portfolio volatility is a function of the covariance or correlation,  $\rho$ , of its component revenue sources (Gentry and Ladd 1994; White 1983).<sup>3</sup> Table 1, which shows two equally weighted revenue sources—the income tax and the alcohol tax—illustrates this basic idea.

Lacking further information about the economy in the coming year, we would assume that the likelihood of each of the five possible states that could occur is equal to its historical rate of occurrence or frequency. For each revenue source, these states are associated with an average year-on-year growth rate. With this information, we can calculate the expected growth and volatility,  $\sigma$ , of each of the revenue sources and of the portfolio as a whole. Expected growth is the weighted average of the growth rates (summed over the possible states of nature), or 4 percent. In contrast, the volatility of the portfolio,  $\sigma_{\pi} = 3.1$  percent, is much less than the volatility of either the income tax (13.4 percent) or the income and alcohol taxes combined (8.9 percent). It is less even than the volatility of the alcohol tax alone (4.4 percent).<sup>4</sup>

Some of the more remarkable implications of portfolio theory are:

**Table 1** An Illustrative, Two-Tax Portfolio

	<i>Prob.</i>	<i>Income</i> %	<i>Alcohol</i> %	<i>Portfolio</i> %
<b>Recession</b>	<b>0.10</b>	<b>-22.0</b>	<b>8.0</b>	<b>-7.0</b>
<b>Below Average</b>	<b>0.20</b>	<b>-2.0</b>	<b>4.0</b>	<b>1.0</b>
<b>Average</b>	<b>0.40</b>	<b>10.0</b>	<b>0.0</b>	<b>5.0</b>
<b>Above Average</b>	<b>0.20</b>	<b>18.0</b>	<b>-4.0</b>	<b>7.0</b>
<b>Boom</b>	<b>0.10</b>	<b>30.0</b>	<b>-8.0</b>	<b>11.0</b>
<b>Expected Growth</b>		<b>8.0</b>	<b>0</b>	<b>4.0</b>

- Average volatility will usually be reduced by adding tax sources, except where the two taxes are perfectly correlated (i.e., the coefficient of correlation = +1.0).
- A two-tax portfolio could in theory be combined to eliminate revenue volatility completely, but only if the coefficient of correlation = -1.0 and the two taxes were weighted equally. Unfortunately, there are no such tax types. The closest we come to one is the lottery, with a coefficient of correlation between -0.08 and -0.20.<sup>5</sup>
- In general, tax sources have coefficients of correlation that average about 0.65, so adding taxes to the portfolio tends to reduce but not eliminate volatility.
- Only if we look at efficient tax portfolios is there a necessary tradeoff between stability and growth. Moreover, it is possible to construct an efficient growth frontier showing this tradeoff. All one needs is information on the covariance of the growth rates of each of the different tax types and designs obtained in the different states.

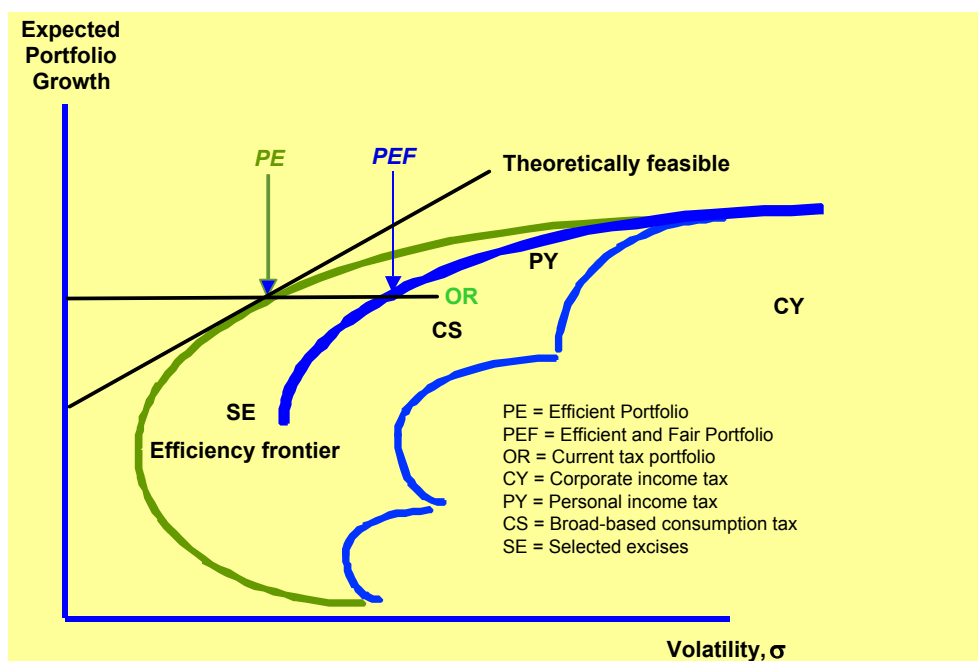
Assessing a tax portfolio is actually fairly easy. Instead of calculating the covariances of each pair of possible tax designs and using that information to laboriously construct an efficient frontier, the analyst need only calculate the relationship of each tax type to an appropriate macroeconomic aggregate such as gross domestic product or state product (whichever is the single most important influence on its revenue growth). If the growth of a tax type is more volatile than movements in the economy, that tax type will make the portfolio more variable than it would have been otherwise. If the tax type is less volatile, it will make the portfolio less risky. The volatility of the portfolio as a

whole is simply the weighted average of these relationships (Gentry and Ladd 1994).<sup>6</sup>

Figure 1 shows various tax portfolio types in risk/growth space. It also shows reasonable combinations of revenue sources and feasible weights that would give Oregon the same real rate of revenue growth that it currently enjoys, the portfolio volatility ( $\sigma_p$ ) for each combination and set of weights, and the minimum, denoted PE (efficient portfolio). This analysis suggests that tax portfolio diversification could reduce Oregon's revenue volatility ( $\sigma$ ) by more than 40 percent (from 7.6 to 4.4 percent) without substantially reducing revenue growth. The problem with this solution is that, while it would significantly reduce revenue volatility, it would do so at the expense of tax fairness. Shifting from Oregon's existing tax portfolio to the efficient portfolio would reduce the cross-sectional income elasticity of Oregon's tax structure by 25 to 35 percent, converting it from moderately progressive to slightly regressive.

This result did not really catch us unaware, since the efficient tax portfolio included sales taxes and greatly increased liquor and motor fuel taxes, but not corporate income taxes. It also cut personal income tax rates in half.<sup>7</sup> Following the approach formulated by Gentry and Ladd (1994), we replicated our search for efficient tax portfolios, this time constraining the weighted average of our synthetic portfolios to cross-sectional income elasticities equal to or greater than 1.25, the mean of Oregon's current tax portfolio. This is also its current rate of revenue growth. Unfortunately, a portfolio that would be both efficient and equally fair, would not significantly reduce tax volatility. This point is denoted PEF (efficient and fair portfolio) in figure 1.<sup>8</sup>

**Figure 1** Efficient and Equitable Tax Portfolios



**Lesson 3:** *Volatility in revenue growth can be significantly reduced via a well-designed portfolio of tax types.*

**Lesson 4:** *Once an efficient frontier has been identified, changes in the portfolio of tax types to increase tax equity also will increase volatility in revenue growth.*

**Lesson 5:** *Even the best designed tax portfolio would not eliminate all volatility in revenue growth.*

Lacking negatively covariant tax types, the best tax-portfolio designers can hope to do is to eliminate the unsystematic or random portion of the variation in revenue growth. The systematic portion would remain.<sup>9</sup> Moreover, portfolio analysis tells us that we need a lot of tax types to eliminate all unsystematic variance in revenue growth. We may not have enough tax types to work with.<sup>10</sup>

In the next section of this essay we will discuss mechanisms for managing systematic volatility.

### Managing Systematic Risks with Hedges and Buffers

One way to reduce volatility in revenue growth is to offset it with a revenue flow of equal size and opposite volatility. This is called hedging. As we have seen, if we could find two tax types that produced revenue flows of the same size and that were perfectly but inversely correlated with each other, we could eliminate all volatility in revenue growth. Unfortunately, there are no such tax types. Nevertheless, it is possible to design hedges against the systematic component of revenue volatility.

### Using financial derivatives to hedge against systematic revenue volatility.

Hedging with futures and options contracts is the cheapest, most direct way to deal with systematic volatility in revenue growth, at least in theory (Swidler, Buttimer, and Shaw 1999). Indeed, many of the expedients currently used by governments to offset systematic volatility in revenue growth are conceptually analogous to the use of futures and options contracts. For example, governments

issue variable rate bonds precisely because of the inverse correlation between the prime rate and revenue flows, which means that they work like futures contracts. Governments could go even further in this direction by directly indexing debt service on their bonds to revenue growth. And, of course, many government debt issues feature an embedded *call* option, which gives the issuer the right, but not the obligation, to *buy* them back at a fixed price at a future date (and the issuer will usually do so if rates fall sufficiently). As we will show, participating in self-insurance pools and investing for a rainy day are both also similar to options, at least conceptually.

Governments could significantly offset systematic revenue risk and thereby stabilize revenue growth by *selling* futures contracts on an index based upon the single most important macroeconomic factor influencing their revenue growth. In this case, the underlying asset values would be covariant with the systematic portion of the government's revenue flows. This is simply a fancy way of saying that the profit or loss from the sale of these futures contracts would vary inversely with the systematic component of the variance in government revenue growth.<sup>11</sup>

However, as Cristoph Hinkelmann and Steve Swidler (2005, 129) observe: "A futures contract is...a zero sum game in which the profits of one party equal the losses of the counterparty." Using futures contracts to hedge revenue flows means sacrificing higher-than-expected revenue flows and not just avoiding revenue shortfalls. If governments wanted only to protect themselves against revenue shortfalls (keeping high revenue inflows for themselves), they would have to purchase "put" options on the covariant financial asset. A *put* gives the buyer the right, but not the obligation, to *sell* the underlying financial asset for a fixed price,

called the strike price. Of course, a government would exercise its option to sell the asset only if its spot price fell below the strike price. Unfortunately, options aren't free. Governments would have to pay hefty premiums to purchase enough put options to provide meaningful insurance against unwanted revenue shortfalls. Moreover, the greater the volatility of a government's tax portfolio relative to the volatility of the index, the larger the hedge needed.

Currently, thanks largely to Yale professor Robert Shiller (2003), there is a movement afoot to create hedging instruments based on indices of macroeconomic aggregates. Goldman Sachs and Deutsche Bank already offer derivatives on non-farm payroll and initial jobless claims. The Chicago Board of Trade offers futures and options contracts based on indices such as gross national product and personal income. None of these indices would represent a perfect hedge (a one-for-one offset for gains or losses) against revenue volatility. But they are pretty good. According to Hinkelmann and Swidler (2004, 2005), derivatives based on personal income (either futures or put options) could reduce systematic revenue volatility by 60 to 80 percent in about twenty states, including Massachusetts, New York, Ohio, and Pennsylvania.

***Lesson 6: Hedging with futures and options contracts is potentially the cheapest, most direct way to deal with systematic volatility in revenue growth.***

Nevertheless, it seems unlikely that governments will embrace the use of futures and options contracts to hedge against systematic revenue volatility any time soon, both for political and technical reasons. For the time being, their greatest advantage derives from the fact that they can be fairly easily priced. Hence, we can estimate that the pre-

mium on a one-year put option would typically cost about \$8 to \$10 million for each \$1 billion hedged. This figure can be used as a benchmark against which alternative means of coping with revenue volatility can be assessed.

**Coping with systematic revenue volatility using rainy day funds.** The two most common suggestions for dealing with revenue volatility are putting money aside for revenue shortfalls in what is commonly called a rainy day fund and participating in a multi-government risk pool, with specific experience ratings created to reflect revenue volatility. The former suggestion implies a program of self-insurance, while the latter is comparable to buying an insurance policy, with the annual contribution playing the part of the insurance premium. Insurance policies are little more than put options, since they give the holder the right to exercise them only under specified conditions.

The problems raised by a policy of self-insurance against revenue shortfalls are twofold. The first problem is estimating the size of the rainy day fund needed to stabilize spending, given systematic variations in revenue flows. The second problem arises in formulating a contribution or savings rule to follow to achieve the desired fund size. To address these issues, Gary Wagner and Erick Elder (2004) used a Markov-switching model to estimate real per-capita personal income for each state during booms and busts, as well as the probability of switching from economic expansion to contraction and back again. Based on these results (together with the questionable assumption that state revenues vary directly with personal income), they computed the savings rate needed to buffer state governments against unrequited revenue shortfalls. They found that to provide a 90 percent buffer against cash shortfalls, the required contribution rate

was, on average, \$19 million per \$1 billion of revenue. Ten states needed contribution rates of less than \$10 million per million, with Kansas requiring none. Eight states needed contribution rates of more than \$30 million per billion, with Wyoming topping the list at \$45 million per billion.<sup>12</sup>

Perhaps the most interesting attempt to estimate the savings needed to insure against revenue shortfalls looks at local governments in Minnesota (Kriz 2002). According to Ken Kriz, the level of savings needed depends upon several factors: average revenue growth, revenue volatility, average return on investments, volatility of investment returns, and the desired rate of revenue growth. To compute this value, Kriz assumed that revenues and investment returns could be modeled by a Markov process called geometric Brownian motion. This is a type of stochastic process in which the distribution of future values of a financial variable, conditional on current and past values, is identical to the distribution of future values, conditional on the current value alone. Growth increments are independent of one another, and the variance of the change in the process grows linearly with its time horizon. This process is also called a random walk. Kriz used a Monte Carlo simulation to replicate this process and thereby to estimate the level of savings needed to sustain a given rate of expenditure growth. As Kriz notes, Monte Carlo simulation is widely used to assess financial strategies in the face of uncertainty, although it is rarely used in public finance. Kriz found that if a jurisdiction wished "to sustain a three percent expenditure growth rate with a 75 percent confidence level, it would need savings equal to 91 percent of total revenues" (Kriz 2002, 5).

**Rainy day fund pools as a means of coping with revenue volatility.** The recognition that most governments fail to adopt contri-

bution and withdrawal rules adequate to safeguard their rainy day funds led Holcombe and Sobel (1997) to suggest that state governments establish a pool that would operate independently of its members.<sup>13</sup> They plausibly argued that an arm's length relationship would reduce local state pressure to spend cash reserves whenever they reached significant levels, better even than a statutory spending rule. Holcombe and Sobel further noted that clear rules governing contributions and withdrawals would improve state credit ratings and thereby reduce capital financing costs for states. Finally, they argued that by pooling their funds, the states could significantly reduce the amount of money each would have to contribute to achieve a given level of stability.

This last conclusion logically follows from treating the determination of cash balances as an inventory problem. The standard formulation of this problem under uncertainty holds that the minimum inventory needed to buffer against shortages (a given percentage of the time) is a function of the square root of the size of the pool. Holcombe and Sobel did not rest their argument on this formulation alone, however. Their conclusions reflected an insightful application of portfolio analysis. They assessed the covariance of the revenue yields of various tax types with macroeconomic aggregates. Their findings showed that the collective or pooled state variance was substantially less than the sum of the individual states. Based on their calculations, state participation in a savings pool would average about \$15 million for each \$1 billion of revenue collected, a cost about 15 percent less than self-insurance.

Subsequently, Richard Mattoon (2004) of the Federal Reserve Bank of Chicago designed a national state rainy day fund modeled on the unemployment compensation trust fund, a widely used countercyclical risk

management tool. Mattoon proposed the creation of an experience ratings system that would trigger differential fund contributions for each state and permit borrowing from the national fund. Borrowing states would be charged interest for the use of their own funds. Mattoon also simulated fund performance under differing rules governing contributions and withdrawals.

***Lesson 7: Self-insurance and risk pooling are like put options. A rainy day fund is a form of self-insurance.***

***Lesson 8: A rainy day fund large enough to prevent all revenue shortfalls where systematic revenue volatility is high would be very costly.***

***Lesson 9: Risk pooling would reduce those costs somewhat.***

### **Using Stochastic Process Analysis to Formulate Spending Rules**

Consumption or expenditure smoothing is the last approach we will consider to managing revenue volatility. Under consumption smoothing, governments would use savings and/or borrowing to smooth out consumption over time. Formally put, consumption smoothing implies present-value balance, which means that the present value of a state's projected revenues plus its net financial assets (assets minus liabilities) are equal to or greater than the present value of its projected outlays (Baker, Besendorfer, and Kotlikoff 2002). Where present-value balance is violated, permanent reductions in spending or increases in taxes are unavoidable.

Hence, the problem faced by budgeters is identifying the maximum rate of growth in spending level consistent with present-value balance, given the government's existing revenue structure and volatility.<sup>14</sup>

***Lesson 10: Governments should balance budgets in a present-value sense, using savings and debt to smooth spending.***

**A first approximation.** As a first approximation, Schunk and Woodward (2005; also see Hou 2006) shorten and cloud the question's time horizon. They describe fiscal sustainability not in terms of present-value balance, but as *sufficient revenue to meet a state's fiscal obligations over the course of the business cycle*. Both definitions imply, however, that governments could achieve balance, at least in theory, by offsetting revenue shortfalls in bad times with revenue windfalls in good ones. That is, they could offset deficits with surpluses. Schunk and Woodward conclude that the answer to the problem of sustainability lies in stabilizing spending growth through target budgeting.

To this end, they propose a spending rule for state governments in which spending is allowed to increase no faster than the sum of population growth, plus inflation, plus one percent real growth. Revenue in excess of this amount would be partly diverted to a stabilization (or rainy day) fund, with the rest returned to the taxpayers. They then tested this model using aggregate spending and revenue data from the 50 states for the period 1992-2002. They found that, with a modest portion of surplus revenues partially invested in a rainy day fund, their spending rule resulted in "stable growth of state budgets throughout the recession and sluggish recovery of the early 2000s" (2005, 105). Looking at California and South Carolina individually, they obtained similar results. California diverged from a sustainable path as early as 1996 or 1997, but would have been fine if it had merely practiced a little spending restraint over the next four or five years. South Carolina would have survived intact had it followed their rule. However, it

would have needed to put a higher portion of its surplus revenues into a rainy day fund than California, and that fund would have been almost completely depleted by 2004.

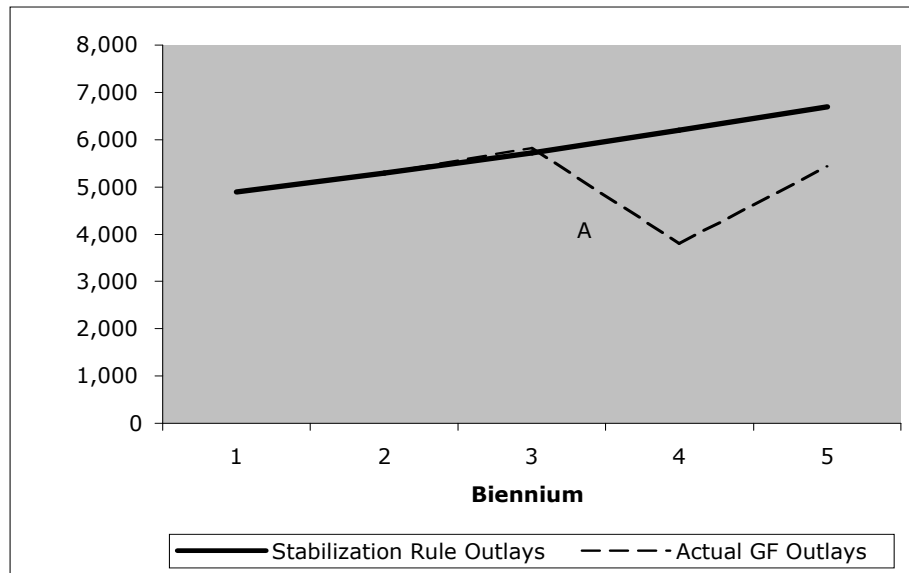
Schunk and Woodward conclude:

This spending rule has the effect of forcing fiscal discipline on state governments, not for the purpose of cutting the size of government, but for the explicit goal of providing stability over the business cycle. This stability is a virtue because it provides a benchmark for state budget writers. It is a rule that governs how much money can be spent while still leaving it up to the discretion of lawmakers to decide how to allocate these funds. (2005, 119)

We think this is a fairly reasonable, practical answer to the budget problem. It is not without weaknesses, however. One weakness is that it is an arbitrary, one-size-fits-all solution. As such, it ignores both the unpredictability of the business cycle and variations in revenue codes. This means that differences in growth trends and revenue volatility are ignored as well. Moreover, when we applied Schunk and Woodward's spending rule to the Oregon case, it had no effect whatsoever on the consequent instability. Figure 2 shows that under their stabilization rule, actual spending tracked allowable expenditures almost perfectly during the first three biennia of the decade. But actual spending then fell way below the stabilization rule and stayed there for the remainder of the decade.<sup>15</sup>

Another problem with Schunk and Woodward's spending rule is that it does not allow for annual adjustment of spending levels on the basis of new information. The utility of using new information to update spending

**Figure 2** Oregon Spending, Actual and Stabilization Rule



plans can be seen by modeling state revenue growth as Wiener processes with drift, estimating the drift coefficient, variance, and mean terminal value of the process by mathematical simulation (in this case, a Monte Carlo simulation).

To simulate the results of a Wiener process, all one needs is a spreadsheet and information on the mean and standard deviations of actual revenue increments. The spreadsheet uses that information to generate future values contingent upon current values. One then runs the simulation several hundred times using a very long time horizon to specify the values of interest. That is basically what we depict in figure 3. In this instance, we assumed a starting point in which revenue and expenditure were set equal at \$100 million. We then applied Schunk and Woodward's spending rule to estimate expenditure growth, and used Oregon's historical data (1963-2003) to calculate the mean and initial variance of revenue growth.

The red line in figure 3 shows the mean cumulative deficit/surplus estimated over 800

runs of the simulation and 100 periods; the upper and lower bounds, the standard deviations; and the black line, one iteration of the 100-period simulation. The reader will note that this simulation suggests that, if Oregon retained all of its revenue and used savings and debt to stabilize expenditure growth, it could sustain spending levels well above those allowed by Schunk and Woodward. If, however, the actual path followed by state revenue growth were as depicted by the 100-period run shown in figure 3, the debt levels incurred during the first 50 years of the process might make state officials very anxious. Consequently, rather than running the process on automatic pilot for the next 99 years, it would make a lot more sense to update the model each year to take account of new information, adjusting planned spending growth accordingly.

Of course, treating revenue growth as a Wiener process implicitly assumes that the detrended process is essentially random noise. Clearly, that is not always the case. Public officials make all kinds of taxing decisions. These decisions affect what we have

**Figure 3** Monte Carlo Simulation Where Revenue Growth is Modeled as a Wiener Process



called noise, as well as drift. Nevertheless, the assumption that the detrended process is essentially random noise allows us to separate the problem of present-value balance (drift) from the problem of cyclical imbalance (noise). Moreover, the Oregon data we used in our revenue growth model is especially amenable to this interpretation, since Oregon's revenue structure has not changed materially over the last 40 years.

**A direct answer.** Schunk and Woodward's approach seems to us like a rather round-about way to address the question of consumption or expenditure smoothing. Why not address it directly? Current savings (net financial assets, which could be either positive or negative), average revenue growth, revenue volatility, average return on investments, and volatility of investment returns are all eminently knowable. Consequently, it is possible to reformulate Kriz's model so there is a single unknown—the maximum rate of growth in the spending level consistent with present-value balance. The mathematically sophisticated reader will note that, if we can then treat revenue and savings

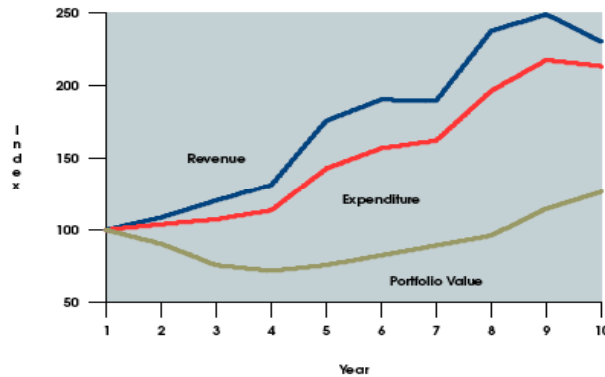
growth as continuous-time, continuous-state stochastic processes, it ought to be possible to calculate a spending rule directly using optimal control theory.

In a recent paper, Dothan and Thompson (2006) do precisely that. They analyze the interaction of government revenues, the investment performance of stabilization accounts, and optimal expenditure levels, where budget balance in a present-value sense must be respected and the jurisdiction uses savings and debt to smooth spending. Then, using fancy math—optimal control theory and martingale methods—they derive time-consistent government-spending rules that depend only on current state variables (see Figure 4).<sup>16</sup>

Their optimal spending growth rule is log-linear in rates of change and linear in time. However, it can be approximated reasonably well by a linear rule in cumulative rates of change. In this case, optimal risk and time-preference-neutral spending growth would be equal to the average rate of revenue growth, less approximately one-half of its

**Figure 4** Indexes of Revenue, Portfolio Value, and Expenditure

The blue line shows an index of revenue, the beige line shows an index of portfolio value, and the red line shows an index of optimal expenditure. All indexes are normalized to 100 in year 1.



variance, adjusted for portfolio growth (see appendix).

By comparing proposed spending levels (including tax cuts and debt service) against the maximum rate of expenditure growth calculated using this rule, one can say whether a specified spending level is sustainable.

Moreover, since the Dothan and Thompson formulation accounts for a government's net financial assets and the returns on those assets (positive or negative), as well as the government's tax structure, it can assess a government's savings and borrowing policies and not just its spending proposals.

***Lesson 11: Optimal rates of expenditure growth can be identified with considerable precision from one year to the next.***

While the mathematics needed to identify a time-consistent spending rule are technically demanding, those needed to calculate expenditure growth rates using this rule are not. Fortunately, one does not have to understand how a rule was derived to use it. Besides, Dothan and Thompson provide an easy way to calculate an approximation of their rule, which should be sufficient to meet the needs of most budget analysts. Investment analysts and bond raters would proba-

bly prefer the more formal version of their optimal spending rule, despite its greater mathematical difficulty. An objective measure of fiscal sustainability would allow them to assess a government borrower's ability to service its debt obligations and, therefore, to grade and price that debt more accurately—in good times as well as bad. Their use of the Dothan and Thompson optimal-spending rule to grade and price government debt would, in turn, provide a powerful incentive for governments to act in accordance with its prescriptions.

In the last few years, a parallel literature has been elaborated to assess a government's ability to maintain its current policies while remaining solvent.<sup>17</sup> A formal statement of the problem appears in the International Monetary Fund's (IMF) *Sustainability Assessments—Review of Application and Methodological Refinements* (2003). The report asks: How much debt can a government assume without violating present-value balance? Its conclusion looks a lot like Schunk and Woodward's. While the IMF proposes several tests that could help analysts evaluate the sensitivity of these guidelines to foreign exchange risk and revenue volatility, its approach (like Schunk and Woodward's) is essentially a one-size-fits-all solution.

Two additional papers bring revenue volatility directly into the picture. Mendoza and Oviedo (2004) explain how government debt limits can be calculated assuming stochastic processes and using Monte Carlo simulation methods similar to those outlined above. Xu and Ghezzi (2003) price government debt and calculate continuous-time, fair-yield spreads and default probabilities using optimal control theory. Both approaches must make somewhat arbitrary assumptions about the willingness of governments to cut spending. This means that while both approaches are concerned with maximum sustainable debt levels, they look a lot like Kriz's approach to estimating optimal savings, at least analytically. Debt/savings and financial liabilities/assets are simply moved from one side of the present-value equation to the other. Consequently, we think it makes more sense to formulate the problem of fiscal sustainability in terms of a government's spending levels. This approach provides the same information and is analytically/mathematically neater because it is more direct.

### Conclusion

Present-value balance is a fail-safe approach to consumption smoothing. Consequently, the main problem faced by state budgeters is to identify the maximum growth rate in spending from one year to the next that is consistent with present-value balance. Attention also should be given to managing unsystematic revenue volatility and hedging systematic revenue volatility.

In her magisterial review of the literature on state budgeting and finance, Irene Rubin wrote:

Much of the literature on state-level budgeting follows the states' adaptation to and responses to cycles of boom and bust in the economy, including preven-

tion (building up reserve funds that can be used in time of recession), temporizing (using delaying tactics to tide the state over until the economy improves), and balancing (increasing revenue and/or decreasing spending)... What would be useful here is an index of prevention and perhaps a second and related one of preparation for recessions. (2005, 47-8, 65)

Rubin has it right, with one *caveat*. Imbalances are not the result of recessions, but of cyclical and random changes in revenues *and* outlays. Indeed, unsustainable spending inspired by the length and magnitude of the Clinton-era boom offers an answer to how the relatively mild and short-lived 2001 recession could have led to such big fiscal problems for the states (Boyd 2000; Vasché and Williams 2005). That caveat aside, what we have done here is provide the indices of preparation and prevention called for by Professor Rubin. Using them will help public officials produce a good outcome no matter what the economy does.

### Notes

1. For our purposes, a soft budget constraint refers to a fiscal environment in which decentralized (regional, state, or local) levels of government can expect another (typically a higher) level of government to rescue them from fiscal distress or where they can force central banks to buy their bonds. Where such prospects do not exist, governmental decision-makers are said to confront a "hard budget constraint." This is one of the great strengths of American fiscal federalism (Inman 2003). One of its weaknesses is the excessive focus of state and local governments on the general fund rather than on a more comprehensive all funds basis.
2. This essay is concerned with simple mean-variance analysis; we ignore more complex non-normal, skewed, or kurtotic distributions. These are of increasing importance in financial theory, but the kind of unknowns that would render moot the analysis described here and would probably also confront governments with challenges that would make the question of fiscal balance hardly worth mentioning.

3. This is, of course, merely a special application of ideas formulated in corporate finance having to do with risk and return (see Lintner 1965; Markowitz 1952; Sharpe 1964).
4. These values were selected for illustrative purposes and do not necessarily reflect anything real. Moreover, in the example, the two tax types are equally weighted (i.e., they produced the same revenue last year). Unequal weights complicate this calculation. Consequently, we use information about the covariance ( $\rho$ ) of the components of the portfolio to calculate portfolio standard deviations. If, for example, the more stable tax source A had a weight of 0.3 and a  $\sigma$  of 0.2, the less stable tax source B had a weight of 0.7 and a  $\sigma$  of 0.4, and their  $\rho$  was 0.4, then:

$$\begin{aligned}\sigma_p &= \sqrt{W_A^2 \sigma_A^2 + (1 - W_A)^2 \sigma_B^2 + 2W_A(1 - W_A) \rho_{AB} \sigma_A \sigma_B} \\ &= \sqrt{0.3^2(0.2^2) + 0.7^2(0.4^2) + 2(0.3)(0.7)(0.4)(0.2)(0.4)} \\ &= 0.309\end{aligned}$$

5. Thirty-one states had both a state income tax and a fully operational lottery for fiscal years 2000 thru 2005. We looked at the annual percentage change in both personal income tax revenues and net lottery income, which is pretty close to net revenue. For all 31 states, we get a correlation coefficient of -0.16 on percentage change for a sample of 155 observations. So, while lottery transfers are quite small (averaging somewhere on the order of 3 percent of state own-source revenues), they do help offset the volatility of personal income taxes.
6. This is, of course, merely a special application of William Sharpe's capital asset pricing model or CAP-M, pronounced CAP-EM. The model starts with the assumption that movements in financial variables are related through common relationships with an underlying dominant factor, which for our purposes is growth in the underlying economy (Sharpe 1964). In CAP-M, the relationship between the financial variable and the underlying dominant factor is called the financial variable's beta. This is named after the normalized coefficient in the OLS formulation of the regression model used to estimate the relationship. Furthermore, once an efficient portfolio has been identified, we can resurrect the traditional tradeoff between stability and revenue growth, but on a far firmer analytical footing.
7. This tax portfolio is nevertheless quite attractive on several dimensions. Because it would entail substantially lower marginal deadweight losses than Oregon's current tax portfolio, it also would be allocatively more efficient than Oregon's existing tax portfolio (see Diewert, Lawrence, and Thompson 1998).

8. It should be possible to design a broad-based consumption tax (e.g., a value-added tax with direct income-contingent rebates to taxpayers) that would be as progressive as Oregon's personal income tax. This means that the theoretically feasible efficient frontier is probably much closer to the efficient frontier than shown in Figure 1. Figure 1 reflects only the tax types actually employed by the states and not what might be theoretically possible.
9. In our opinion, the best normative reasons for utilizing a broad portfolio of tax types are increasing horizontal equity in tax incidence (where the relevant income measure is permanent income, an outcome that would be approximated if all unsystematic sources of variation in revenue growth were eliminated) and reducing the deadweight burden of taxation via a broadening of the tax base (Diewert, Lawrence, and Thompson 1998). From this perspective, reducing volatility in revenue growth is a secondary consideration.
10. Including local revenue sources in state government portfolios would help. It also might reduce the propensity of states to plunder local fund balances whenever they get into trouble (Kriz 2002, 1).
11. This could be accomplished in theory by selling futures contracts on state revenue, but problems of moral hazard and, perhaps, adverse selection would probably make the design and operation of such markets prohibitively costly.
12. Gates et al. (2005) use a Monte Carlo simulation of a simple inventory model to estimate the rainy day fund Oregon would need to buffer itself against unwanted spending shortfalls. In their analysis of Utah's rainy day fund, Gary Cornia and Ray Nelson (2003) utilize a value at risk (VAR) model rather than an inventory model. VAR identifies the worst loss over a target horizon, with a given level of confidence. As such, it is widely used in the risk management literature. Cornia and Nelson used this approach because the software for it was readily available and because they wanted to develop a measure of risk over the business cycle for Utah's unique tax structure. Analytically speaking, however, what they did is not significantly different from what was done by Gates et al. or Kriz. But by decomposing the variance in Utah's cash flows into systematic and unsystematic components, their work inspired our attempt to show how a variety of risk management tools fit together to address the revenue volatility problem.
13. Mattoon argues that "a quasi-governmental agency created by the states would be the logical organization to administer the fund. The agency would need to be autonomous enough to enforce rainy day fund rules and to have sufficient expertise to adjust rainy day fund structure to reflect emerg-

ing conditions. If specific experience ratings were created to reflect state revenue and expenditure volatility, the agency would need to have the staff expertise to calculate annual experience ratings. The agency would need to function as an independent third party administrator" (2004, 18).

14. Expenditure or consumption smoothing was adumbrated in two papers published by Ramsey (1927, 1928), as was the solution outlined below.
15. Arguably, this result was not inconsistent with Schunk and Woodward's analysis. Oregon has long had a legislatively enacted expenditure limit that is almost identical to Schunk and Woodward's spending rule. The difference is that Oregon sent the entire surplus revenues back to the taxpayers—almost none was set aside for a rainy day. When the rainy day came, Oregon had no money (Thompson and Green 2004).
16. Dothan and Thompson's results differ from most other, largely inconclusive attempts in the literature to use optimal control theory to derive optimal spending rules for national governments. This happens because the existing revenue structure is taken as a given and because the effect of government taxes and spending on the rate of economic development is ignored. By contrast, most optimal growth/tax theorists seek to determine how to adjust fiscal and monetary policy to produce the mix of savings, investment, and consumption that is consistent with the highest sustainable rate of real economic growth. This is a very hard question to answer. It is not surprising that the answers are not conclusive.
17. For a thorough assessment of this literature, see Burnside 2004. For purposes of comparison and contrast, see Denison, Hackbart, and Moody 2005.

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## Appendix: Optimal Budgets and Present Values

Merton (1971) used dynamic programming to solve an optimal consumption and portfolio problem for an investor. Dothan and Thompson (2006) exploit an analogy between government expenditures and investors' consumption to present a generalization of Merton's problem, where government revenue is a Weiner process. That is, they

assume that (1) government's revenue stream,  $y_t$ , is exogenous to the model and (2) that its dynamics over very short time intervals,  $dt$ , are (a) the sum of constant growth at rate  $\mu$  and a random shock represented by an increment of Brownian motion,  $B_t$ , and (b) a continuous-time symmetric random walk. The increments in the random walk have normal distribution, with zero mean and variance  $t$ , scaled by a volatility parameter  $\sigma$ .

The specification of revenue dynamics in equation (1) implies the revenue process in equation (2). The random revenue process in equation (2) is called exponential Brownian motion, and it is characterized by an infinite number of potential sample paths with intermittent periods of exponential growth and decay. At any future time, the distribution of exponential Brownian motion is log-normal.

The budget policymaker's problem is to determine the endogenous optimal expenditure process,  $x_t$ , and the optimal size of a budget reserve account,  $H_t$ . Dothan and Thompson further assume that policymakers have a utility function over the expenditure stream that is discounted by the combined value of population growth and inflation. They denote this combined growth rate by  $\alpha$ , constant relative risk aversion  $\gamma$ , and rate-of-time preference  $\phi$ . They denote the market price of risk by  $\theta$  and the risk-adjusted capi-

talization rate of revenue by  $k$ , and they assume that the reserve account is invested in bonds paying interest rate  $r$ . Policymakers maximize their expected utility subject to a present-value constraint. That constraint equates the risk-adjusted present value of future expenditures with the value of the reserve account, plus the risk-adjusted present value of future revenues.

The solution to this problem is given in equation (3). The first equation in (3) says that optimal expenditure is a constant fraction of wealth  $V$ , defined as the value of the reserve account, plus the risk-adjusted present value of future revenues. The second equation in (3) delivers a representation of optimal expenditure as a power function of revenue multiplied by a deterministic growth factor at a constant rate  $\beta$ . The exponent of the power function is the elasticity of optimal expenditure with respect to revenue. The third equation in (3) describes the optimal value of the reserve account as a function of revenue  $y_t$  and optimal expenditure  $x_t$ . Hence, over time, optimal spending is an exponential Brownian motion.

Because the distribution of optimal spending rate is log-normal, and from the properties of a log-normal distribution, it follows that optimal spending growth is  $\lambda + \omega^2/2$ , and that it may be positive or negative depending on model parameters.

## List of Equations

$$\frac{dy_t}{y_t} = \mu dt + \sigma dB_t \quad (1)$$

$$y_t = y_0 \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right] \quad (2)$$

$$\begin{aligned}
 x_t &= \nu \left( H_t + \frac{y_t}{k - \mu} \right) \\
 x_t &= x_0 \left( \frac{y_t}{y_0} \right)^{\frac{\theta}{\gamma\sigma}} \exp(\beta t) \\
 H_t &= \left( H_0 + \frac{y_0}{k - \mu} \right) \frac{x_t}{x_0} - \frac{y_t}{k - \mu}
 \end{aligned} \tag{3}$$

$$\frac{x_t}{y_t} = \frac{x_0}{y_0} \exp(\lambda t + \omega B_t) \tag{4}$$